

The Structure of Step Fuzzy Subgroups*

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Abstract The concept of step fuzzy subgroups is introduced and the relations between properties of groups and step fuzzy subgroups are studied.

Key words Fuzzy Subgroup; T- norm; Step fuzzy subgroups; Step normal fuzzy subgroup

Introduction

Anthony and Sherwood^[1] proposed a definition of fuzzy subgroup of a group G With respect to (wrt) a t- norm T .

In this paper we introduce the notion of a step fuzzy subgroup of G wrt T and discuss the following two problems:

- (1)The structure of a step fuzzy subgroups,
- (2)the structure of a step normal fuzzy subgroups.

These discussions reveal the relations between properties of group and step fuzzy subgroups.

1 Preliminaries

Definition 1.1 Let X be (an ordinary) set. A fuzzy subset of a set X is a function $\mu: X \rightarrow [0, 1]$ with $\mu(x) \in [0, 1]$ as the grade of membership.

Definition 1.2 By a t- norm T , We mean a function $T:$

$[0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying, for each $x, y, z \in [0, 1]$,

- (1) $T(x, 1) = x$,
- (2) $T(x, y) \leq T(x, z)$ if $y \leq z$,
- (3) $T(x, y) = T(y, x)$,
- (4) $T(x, T(y, z)) = T(T(x, y), z)$.

Throrem 1.3 (see [2]) $T(0, a) = 0$;

- (2) $T(a, b) \leq \min(a, b) = a \wedge b$, for all $a, b \in [0, 1]$.

Definition 1.4 Given two t- norms T_1 and T_2 , we may say that T_1 is stronger than T_2 , written as $T_1 \geq T_2$, if

$$T_1(x, y) \geq T_2(x, y) \quad \text{for all } x, y \in [0, 1].$$

Consider a particular t- norm T_0 : $T_0(a, b) = a \wedge b$ for all $a, b \in [0, 1]$. By theorem 1.3

(2). we have

$$T \leq T_0 \text{ for all } t\text{-norm } T.$$

Definition 1.5 Let G be a group. A function $\mu: G \rightarrow [0, 1]$ is a fuzzy subgroup of G with respect to (wrt) a t -norm T if for all $x, y \in G$,

$$(1) \mu(xy) \geq T(\mu(x), \mu(y));$$

(2) $\mu(x^{-1}) = \mu(x)$ where x^{-1} is the inverse of x in G . Furthermore, a fuzzy subgroup of G with respect to T is called a normal fuzzy subgroup of G wrt T if

$$\mu(xy) = \mu(yx) \text{ for all } x, y \in G.$$

The α -level set of μ , denoted by μ_α , is defined by

$$\mu_\alpha = \{x \mid \mu(x) \geq \alpha\} \quad \text{if } \alpha \in [0, 1].$$

2 Step fuzzy subgroup

Let G be a group and T be a t -norm. We have the following

Definition 2.1 Let μ be a fuzzy subgroup of G wrt T . Then μ is called a step fuzzy subgroup of G wrt T if for every $\lambda \in [0, 1]$ μ_λ is a subgroup of G when $\mu_\lambda \neq \emptyset$

Theorem 2.2 Let μ be a fuzzy subgroup of G wrt T . Then μ is a step fuzzy subgroup of G wrt T if and only if

$$\mu(xy) \geq \mu(x) \wedge \mu(y) \quad \text{for all } x, y \in G.$$

Proof. Suppose $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$.

When $\mu_\lambda \neq \emptyset$, $\lambda \in [0, 1]$, let $x, y \in \mu_\lambda$. Then

$$\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y^{-1}) = \mu(x) \wedge \mu(y) \geq \lambda,$$

have $xy^{-1} \in \mu_\lambda$ is a subgroup of G .

Conversely, suppose there are $x, y \in G$ such that

$$\mu(xy) < \mu(x) \wedge \mu(y).$$

Put $\lambda = \min\{\mu(x), \mu(y)\}$, then $x, y \in \mu_\lambda$. Because

$\mu(xy) < \mu(x) \wedge \mu(y) = \lambda$, $xy \notin \mu_\lambda$. Hence, μ_λ is not a subgroup of G .

Consider the t -norm T_0 : $T_0(a, b) = a \wedge b$, for all $(a, b) \in [0, 1] \times [0, 1]$. We get the following corollary

Corollary 2.3 Suppose μ is a fuzzy subgroup of G wrt T .

Then, 1) μ is a step fuzzy subgroup of G wrt T .

2) Furthermore, μ is a step fuzzy subgroup of G wrt T , where T is an arbitrary t -norm.

Proof 1) Because $\mu(xy) \geq T_0(\mu(x), \mu(y)) = \mu(x) \wedge \mu(y)$, for all $x, y \in G$, it follows from theorem 2.2 that μ is a step fuzzy subgroup of G wrt T_0 .

2) This conclusion follows from 1) and that fact: $T_0 \geq T$.

Definition 2.4 Let μ be a step fuzzy subgroup of G wrt T . Suppose $\mu(G)$ is the range of μ :

$$\mu(G) = \{\mu(x) \mid x \in G\}.$$

If the $\mu(G)$ is finite: $\mu(G) = \{c_0, c_1, c_2, \dots, c_m\}$, with $1 \geq c_0 > c_1 > c_2 > \dots > c_m \geq 0$, the integer m is called the length of μ .

If the $\mu(G)$ is countable: $\mu(G) = \{c_0, c_1, c_2, \dots, c_m, \dots\}$ with $1 \geq c_0 > c_1 > c_2 > \dots >$

$c_m \dots \geq 0$, the length of μ is defined by $+\infty$.

Clearly, if G is a finite group, μ is a step fuzzy subgroup of G wrt T , then the length of μ is a finite integer.

Theorem 2.5 Let μ be a step fuzzy subgroup of G wrt T , the length of μ is s . $\mu(G) = \{c_0, c_1, c_2, \dots, c/2\}$, $1 \geq c_0 > c_1 > c_2 > \dots > c/2 \geq 0$,

Suppose $A = \{\mu_\lambda | \lambda \in [0, 1], \mu_\lambda \neq \Phi\}$. Then

$A = \{\mu_{c_1}, \mu_{c_2}, \dots, \mu_{c_s}\}$ and we can get a subgroup series without repetition in G :

$$\{e\} \leq \mu_{c_0} < \mu_{c_1} < \mu_{c_2} < \dots < \mu_{c_s} = G$$

($\mu_{c_i} < \mu_{c_{i+1}}$ indicate that μ_{c_i} is a proper subgroup of $\mu_{c_{i+1}}$).

Proof. Let i be a integer.

If $0 \leq i \leq s$ then there is a $x_i \in G$: $\mu(x_i) = c_i$, hence $\mu_{c_i} \neq \Phi$. By definition 2.1, μ_{c_i} ($i = 0, 1, 2, 3, \dots, s$) is a subgroup of G .

If $1 \leq i < i+1 \leq s$, then, Clearly, $\mu_{c_i} \subseteq \mu_{c_{i+1}}$. Because there is a $x_{i+1} \in G$ such that $\mu(x_{i+1}) = c_{i+1}$, and $c_{i+1} < c_i$, then $x_{i+1} \in \mu_{c_{i+1}}$, $x_{i+1} \notin \mu_{c_i}$. Hence $\mu_{c_i} \neq \mu_{c_{i+1}}$, $\mu_{c_i} < \mu_{c_{i+1}}$.

Because $e = \min \mu(G)$, then for all $x \in G$, $\mu(x) \geq c_s$, thus $\mu_{c_s} \supseteq G$. Hence $\mu_{c_s} = G$.

Therefore, we get sequence:

$$\{e\} \leq \mu_{c_0} < \mu_{c_1} < \mu_{c_2} < \dots < \mu_{c_s} = G$$

For all $\mu_\lambda \in A$. if $c_{i+1} < \lambda < c_i$, $1 \leq i+1 \leq s$, then

$$\mu_\lambda = \{x | x \in G, \mu(x) \geq \lambda\} = \{x | x \in G, \mu(x) \geq c_i\} \cup \{x | x \in G, \lambda \leq \mu(x) < \mu_{c_i}\} = \mu_{c_i} \cup \Phi = \mu_{c_i}.$$

If $\lambda > c_0$, then $\mu_\lambda = \Phi$, $\mu_\lambda \notin A$.

If $\lambda < c_s$, then $\mu_\lambda \supseteq \mu_{c_s} = G$, hence $\mu_\lambda = \mu_{c_s}$.

Therefore, $A \subseteq \{\mu_{c_0}, \mu_{c_1}, \mu_{c_2}, \dots, \mu_{c_s}\}$

Suppose μ is a step fuzzy subgroup of G wrt T , then what is the structure of μ ? We state with certainty:

Theorem 2.6 μ is a step fuzzy subgroup of G wrt T , and the length of μ is m if and only if μ has the following

structure:

$$\mu(x) = \begin{cases} c_0, & x \in N_0 \\ c_1, & x \in N_1 - N_0 \\ c_2, & x \in N_2 - N_1 \\ \dots & \\ c_m, & x \in N_m - N_{m-1}. \end{cases} \quad (1)$$

in which $1 \geq c_0 > c_1 > c_2 > \dots > c_m \geq 0$, N_i ($i = 0, 1, 2, \dots, m$) are subgroups of G and form a subgroup series without repetition in G : $\{e\} \leq N_0 < N_1 < N_2 < \dots < N_m = G$.

Proof. If μ is a step fuzzy subgroup of G wrt T , and the length of μ is m . Let

$$\mu(G) = \{c_0, c_1, c_2, \dots, c_m\}, 1 \geq c_0 > c_1 > c_2 > \dots > c_m \geq 0.$$

Let $B_i = \{x \mid x \in G, \mu(x) = c_i\}$, $i = 1, 2, \dots, m, 0$.

Then, one can express μ as:

$$\mu(x) = \begin{cases} c_0, & x \in B_0 \\ c_1, & x \in B_1 \\ c_2, & x \in B_2 \\ \dots \\ c_m, & x \in B_m. \end{cases} \quad (2)$$

Clearly, $B_0 = \mu_{c_0}$, $B_i = \mu_{c_i} - \mu_{c_{i-1}}$, $i = 1, 2, \dots, m$.

By Theorem 3.5, we have

$$\{e\} \leq \mu_{c_0} < \mu_{c_1} < \mu_{c_2} < \dots < \mu_{c_m} = G$$

Put $\mu_i = N_i$ ($i = 0, 1, 2, \dots, m$) in (2), we get (1).

Conversely, if μ has the structure (1), clearly, μ is a function, $\mu: G \rightarrow [0, 1]$.

1) For all $x, y \in G$.

If $x, y \in N_i - N_{i-1}$, $1 \leq i \leq m$, then $\mu(x) = \mu(y) = c_i$. Because $xy \in N_i$ (for N_i is a subgroup of G),

$$\mu(xy) \geq c_i = \mu(x) \wedge \mu(y).$$

If $x \in N_i - N_{i-1}$, $y \in N_j - N_{j-1}$, $1 \leq i < j \leq m$, then $\mu(x) = c_i > c_j = \mu(y)$, but $N_i \leq N_j$, hence $xy \in N_j$,

$$\mu(xy) \geq c_j = \mu(x) \wedge \mu(y).$$

If $x, y \in N_0$, then $xy \in N_0$, $\mu(xy) = c_0 = \mu(x) \wedge \mu(y)$. Therefore, we have shown:

$$\mu(xy) \geq \mu(x) \wedge \mu(y), \text{ for all } x, y \in G. \quad (*)$$

But for all t -norm T , $\mu(x) \wedge \mu(y) \geq T(\mu(x), \mu(y))$, for all $x, y \in G$.

Hence, $\mu(xy) \geq T(\mu(x), \mu(y))$, for all $x, y \in G$.

2) For all $x \in G$. If $x \in N_i - N_{i-1}$, $1 \leq i \leq m$, then $x^{-1} \in N_i$, hence, $\mu(x^{-1}) \geq c_i = \mu(x)$. If $x \in N_0$, then $x^{-1} \in N_0$, $\mu(x^{-1}) = \mu(x) = c_0$.

Therefore, we have shown: $\mu(x^{-1}) \geq \mu(x)$, for all $x \in G$.

Put $x^{-1} = y$, by the above inequality, we get

$$\mu(x) = \mu(y^{-1}) \geq \mu(y) = \mu(x^{-1}), \text{ hence } \mu(x^{-1}) = \mu(x), \text{ for all } x \in G.$$

Therefore, by the conclusion of 1), 2) we have shown: μ is a fuzzy subgroup of G wrt T . Furthermore, by (*) and theorem 2.2, we get: μ is a step fuzzy subgroup of G wrt T , and the length of μ is m .

In a similar way to the proof of theorem 2.6, we get Theorem 2.7 μ is a step fuzzy subgroup of G wrt T , and the length of μ is $+\infty$ if and only if μ has the following structure:

$$\mu(x) = \begin{cases} c_0, & x \in N_0 \\ c_1, & x \in N_1 - N_0 \\ c_2, & x \in N_2 - N_1 \\ \dots \\ c_m, & x \in N_m - N_{m-1} \end{cases}$$

in which $1 \geq c_0 > c_1 > c_2 > \dots > c_m > \dots \geq 0$, $N_i (i = 1, 2, \dots, m, \dots)$ are subgroups of G and form a sequence of subgroup of G such that each member of the sequence is a proper subgroup of its successor:

$$\{e\} \leq N_0 < N_1 < N_2 < \dots < N_m < \dots \leq G.$$

$$\text{Let } \{e\} \leq N_0 < N_1 < N_2 < \dots < N_m = G \quad (\star)$$

is a subgroup series without repetition in G . The integer m is called the length of the series.

If there is a subgroup series without repetition $(\star)$ in G , its length $m = \max \{\text{lengths of all subgroup series without repetition in } G\}$, then we may say that the series $(\star)$ is a group series of the longest length in G .

Clearly, finite group has a subgroup series of the longest, but infinite cyclic group has no.

The following theorem 2.8 reveals the relations between step fuzzy subgroup and properties of groups.

Theorem 2.8 Group G has a subgroup series of the longest length which length is m if and only if there is a step fuzzy subgroup μ of G wrt T , the length of μ is m , and for any step fuzzy subgroup of G wrt T , its length $\leq m$.

Proof. If G has a subgroup series of the longest length which length is m : $\{e\} = N_0 < N_1 < N_2 < \dots < N_m = G$.

choose $c_i (i = 0, 1, 2, \dots, m) \in [0, 1]$ such that $1 \geq c_0 > c_1 > c_2 > \dots > c_m \geq 0$. Let

$$\mu(x) = \begin{cases} c_0, & x \in N_0 \\ c_1, & x \in N_1 - N_0 \\ c_2, & x \in N_2 - N_1 \\ \dots \\ c_m, & x \in N_m - N_{m-1}. \end{cases}$$

By Theorem 2.6 μ is a step fuzzy subgroup of G wrt T .

Let μ^* is a arbitrary step fuzzy subgroup of G wrt T . If the length of μ^* is t , and $t \geq m + 1$. By Theorem 2.6, from the structure of μ^* , we can get $c_0, c_1, c_2, \dots, c_t \in [0, 1]$ such that $1 \geq c_0 > c_1 > c_2 > \dots > c_t \geq 0$, and a subgroup series without repetition in G : $\{e\} \leq \mu_{c_0} < \mu_{c_1} < \mu_{c_2} < \dots < \mu_{c_t} = G$

which length $t \geq m + 1$. This contradicts the hypothesis.

Conversely, if there is a step fuzzy subgroup of G wrt T , the length of μ is m , and for any step fuzzy subgroup of G wrt T , its length $\leq m$.

Let $\{e\} = N_0 < N_1 < \dots < N_h = G$ is a arbitrary subgroup series without repetition in G .

Put $c_i \in [0, 1] (i = 0, 1, 2, \dots, h)$ such that $1 \geq c_0 > c_1 > c_2 > \dots > c_h$.

Let

$$L(x) = \begin{cases} c_0, & x \in N_0 \\ c_1, & x \in N_1 - N_0 \\ c_2, & x \in N_2 - N_1 \\ \dots \\ c_h, & x \in N_h - N_{h-1}. \end{cases}$$

By Theorem 2.6, L is a step fuzzy subgroup of G wrt T , thus $h \in m$.

Let L^* be a step fuzzy subgroup of G wrt T , and the length of L^* is m , then, by Theorem 2.6, G has a corresponding subgroup series without repetition, it is a subgroup series of longest length in G , and its length is m .

3 Step normal fuzzy subgroups

Definition 3.1 A wtep fuzzy subgroup of G wrt T is called a step normal fuzzy subgroup of G wrt T if it is a normal fuzzy subgroup of G wrt T .

Theorem 3.2 Let L be a step fuzzy subgroup of G wrt T . Then L is a step normal fuzzy subgroup of G wrt T if and only if for all $K \in [0, 1]$, L_K is a normal subgroup of G when $L_K \neq \emptyset$.

Proof. If L is a step normal fuzzy subgroup of G wrt T . Then, for all $K \in [0, 1]$, when $L_K \neq \emptyset$, L_K is a subgroup of G . For all $x, y \in L_K$, one have $L(xy) = L(yx)$, then $L(y^{-1}xy) = L[y^{-1}(xy)] = L[(xy)y^{-1}] = L[x(yy^{-1})] = L(x) \setminus K$. Hence $y^{-1}xy \in L_K$. $L_K \trianglelefteq G$ (indicate that L_K is a normal subgroup of G).

Conversely, if L is a step fuzzy subgroup of G wrt T , and for all $K \in [0, 1]$, $L_K \trianglelefteq G$ when $L_K \neq \emptyset$. Then for all $x, y, z \in G$, put $L(z) = K$. Because $L_K \trianglelefteq G$, $y^{-1}zy \in L_K$ hence

$$L(y^{-1}zy) \setminus L(z) \quad (*)$$

Let $Z = yx$, by $(*)$, we get $L(xy) \setminus L(yx)$.

Because $zyz^{-1} \in L_K$, $L(zyz^{-1}) \setminus L(z) \quad (**)$

Let $z = xy$, by $(**)$, we get $L(yx) \setminus L(xy)$. Thus

$L(xy) = L(yx)$ for all $x, y \in G$.

Therefore, L is a normal fuzzy subgroup of G wrt T .

Theorem 3.3 L is a step normal fuzzy subgroup of G wrt T . and the length of L is m if and only if L has following structure:

$$L(x) = \begin{cases} c_0, & x \in N_0 \\ c_1, & x \in N_1 - N_0 \\ c_2, & x \in N_2 - N_1 \\ \vdots \\ c_m, & x \in N_m - N_{m-1}. \end{cases} \quad (3)$$

in which $1 \setminus c_0 > c_1 > c_2 > \dots > c_m \setminus 0$, $N_i \trianglelefteq G$ ($i = 0, 1, 2, \dots, m$), and N_i ($i = 0, 1, 2, \dots, m$) form a normal series without repetition in G :

$$\{e\} \trianglelefteq N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \dots \trianglelefteq N_m = G.$$

Proof. If L is a step normal fuzzy subgroup of G wrt T , then by definition 3.1, L is a step fuzzy subgroup of G wrt T , hence by Theorem 3.6 L has structure (1).

Because L is a step normal fuzzy subgroup of G wrt T , by Theorem 3.2 one get $N_i = L_{c_i} \trianglelefteq G$, hence L has a structure (3).

Conversely, if L has a structure (3), by Theorem 2.6, L is a step fuzzy subgroup of G wrt T .

For all $x, y \in G$, if $x \in N_t - N_{t-1}$, $t = 1, 2, \dots, m$, because $N_t \leq G$, $xyx^{-1} \in N_t$, but $N_t = L_{c_t}$, $xyx^{-1} \in L_{c_t}$, hence $L_{c_t} \leq G$, $t = 1, 2, \dots, m$. Furthermore, $L_{c_0} = N_0 \leq G$. Hence $L_{c_i} \leq G$, $i = 0, 1, \dots, m$. By Theorem 3.2 and Theorem 2.5 we have shown that L is a step normal fuzzy subgroup of G wrt T , and its length is m .

In a similar way to the proof of Theorem 3.3, we can get Theorem 3.4 L is a step normal fuzzy subgroup of G wrt T , and the length of L is m if and only if L has the following structure:

$$L(x) = \begin{cases} c_0, & x \in N_0 \\ c_1, & x \in N_1 - N_0 \\ c_2, & x \in N_2 - N_1 \\ \vdots \\ c_m, & x \in N_m - N_{m-1} \\ \vdots \end{cases}$$

in which $1 \setminus c_0 > c_1 > c_2 > \dots > c_m > 0$, $N_i (i = 0, 1, 2, \dots, m, \dots) \leq G$, and $\{e\} \in N_0 \leq N_1 \leq N_2 \leq \dots \leq N_m \leq G$.

Just as Theorem 2.8, the following Theorem 3.5 reveals the relations between normal series of G and step normal fuzzy subgroup of G wrt T .

Theorem 3.5 Group G has a principal series which length is m (see [3]) if and only if there is a step normal fuzzy subgroup L of G wrt T , the length of L is m , and for any step normal fuzzy subgroup of G wrt T , its length $\leq m$.

Proof. with the Theorem 3.2, this proof is similar to the proof of Theorem 2.8.

We know that finite group has a principal series and that infinite cyclic group has no. By Theorem 3.5, we get Corollary 3.6. If G is a finite group, then there is some step normal fuzzy subgroup of G wrt T .

Remark. Clearly, a fuzzy subgroup of G wrt T_0 (see definition 1.4) is a step fuzzy subgroup of G wrt T_0 and a normal fuzzy subgroup of G wrt T_0 is a step normal fuzzy subgroup of G wrt T_0 . Therefore, the conclusions of section 3 and section 4 are suitable for fuzzy subgroups of G wrt T_0 and normal fuzzy subgroups of G wrt T_0 .

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阶梯型模糊子群的结构

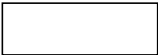
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定义了模糊阶梯型子群, 并研究了阶梯型模糊子群与经典群之间的关系.

群论; 模糊子群; T 模; 阶梯模糊子群

中图资料法分类号 O159; O152.1



/ 高硫铅锌矿电位调控浮选研究与应用0通过鉴定

我校环资系刘如意教授主持的/ 高硫铅锌矿电位调控浮选研究与应用0项目 5 月 5 日在广西金城江市通过技术鉴定. 由中科院、工程院两院院士王淀佐教授为首的专家组对该项目的研究与应用成功给予很高的评价, 认为项目提出的新工艺具有技术先进、流程简单、药剂用量少、分选指标高、环境污染少、经济效益与社会效益显著等优点, 属国内外首创, 在浮选电化化学理论与实践和处理同类矿石的浮选技术上都达到了国际先进水平.

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